Linear Programming (LP): Simplex Search

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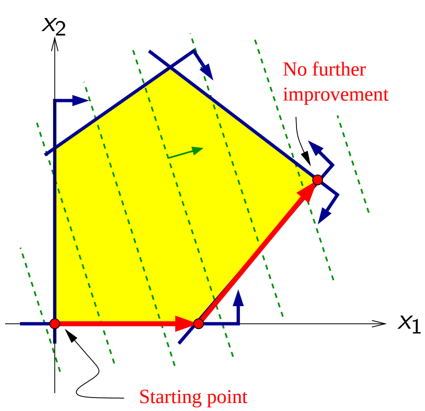
Outline

For additional details, see Rardin (1998), Chapter 5.2-5.8

Simplex: An Extreme-Point Search Algorithm



- Consider *only* adjacent extreme points for improvement direction
- Move along the edge that yields the greatest rate of improvement
- Move until another extreme point has been reached
- Check if further improvement is possible: if 'yes', continue; else, stop

Outstanding Questions:

1. How to characterize extreme feasible points?
2. How to move to an adjacent extreme feasible point?
3. How to start at an extreme feasible point?

Basic Solutions

LP standard form:

$$\min_{\mathbf{x}} \ z \triangleq \mathbf{c}^\mathsf{T}\mathbf{x}$$
$$\text{s.t.} \quad \mathbf{A}\mathbf{x} = \mathbf{b}$$
$$\mathbf{x} \geq \mathbf{0}$$

$x_j \triangleq$ jth decision variable
$c_j \triangleq$ objective function coefficient of x_j
$a_{i,j} \triangleq$ constraint coefficient of x_j in the ith main constraint
$b_i \triangleq$ right-hand side (RHS) constant term of main constraint i
$m \triangleq$ number of main constraints
$n \triangleq$ number of decision variables

Definition:

A basic solution to a linear program in *standard form* is one obtained by fixing *just enough* variables to $= 0$ that the model's equality constraints can be solved *uniquely* for the remaining variable values

- Those variables fixed at zero are called nonbasic variables
- The ones obtained by solving the equalities are called basic variables

Computing Basic Solutions

Class Exercise: Consider the following linear program:

$$\begin{aligned} \min \quad & 9x_1 + 6x_2 \\ \text{s.t.} \quad & 4x_1 + x_2 = 1 \\ & 3x_1 - 2x_2 \leq 8 \\ & x_1 \geq 0, x_2 \leq 0 \end{aligned}$$

Question: Compute the basic solution corresponding to x_1 and x_2 basic.

Existence of Basic Solutions

Can basic solutions be formed by setting *any* collection of nonbasic variables to zero? Not so...

Important Property

A basic solution exists if and only if the system of equality constraints corresponding to basic variables has a unique solution

Math Refresher:

1. How do you calculate the determinant of an m-by-m matrix $\mathbf{A}$?
2. Give a necessary and sufficient condition for the system of m linear equations $\mathbf{Ax} = \mathbf{b}$ to have a unique solution in $\mathbb{R}^m$.

Checking Existence of Basic Solutions

Class Exercise: The following are the constraints of a standard form LP:

$$\begin{aligned} & 4x_1 - 8x_2 - x_3 = 15 \\ & x_1 - 2x_2 = 10 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Question: Do basic solutions exist for the sets of basic variables:

(a) $\mathscr{B} = \{x_1, x_2\}$:

(b) $\mathscr{B} = \{x_1\}$:

(c) $\mathscr{B} = \{x_2, x_3\}$:

Basic Feasible Solutions and Extreme Points

Definition

A **basic feasible solution** to an LP in standard form is a basic solution that satisfies *all* nonnegativity constraints

Class Exercise: Is the solution corresponding to x_2 and x_3 basic in the previous exercise a basic feasible one?

Fundamental Result

The basic feasible solutions of a linear program in standard form are exactly the extreme points of its feasible region

- Since algebraic tests exist to check basic feasible solutions, we now are in a position to characterize extreme points (i.e. potential optimal solutions) algebraically
- This is quite a significant result!

Identifying Basic Feasible Solutions

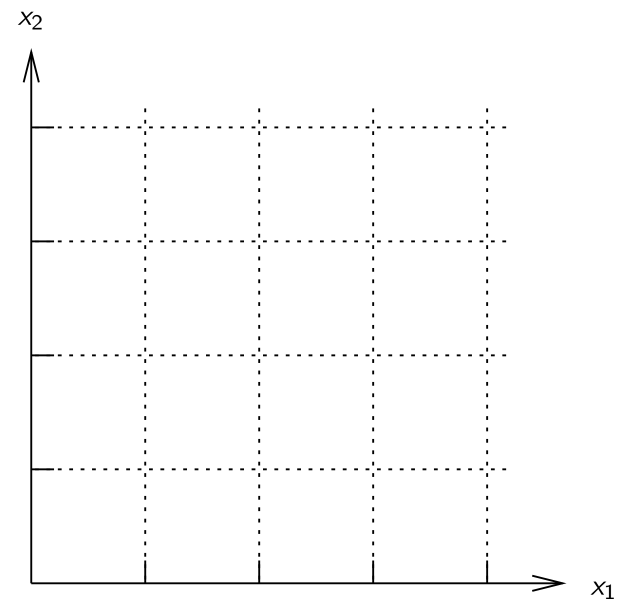
Class Exercise: The following are the constraints of an LP model:

$$
\begin{aligned}
-x_1 + x_2 &\geq 0\\
x_1 &\leq 2\\
x_2 &\leq 3\\
x_1, x_2 &\geq 0
\end{aligned}
$$

Questions:

1. Graph the feasible region and indicate the extreme points
2. Rewrite the LP in standard form by introducing slack variables x_3, x_4 and x_5 in the main constraints 1 to 3, respectively.
3. Compute the basic solutions corresponding to the following sets of basic variables, and determine which are basic feasible solutions: (a) $\mathscr{B}_1 = \{x_3, x_4, x_5\}$; (b) $\mathscr{B}_2 = \{x_1, x_2, x_4\}$; (c) $\mathscr{B}_3 = \{x_1, x_2, x_5\}$. Verify that each basic feasible solution corresponds to an extreme point in the graph. How about basic infeasible solutions?

Identifying Basic Feasible Solutions (cont'd)



Simplex Algorithm: Principles

- The simplex algorithm is a variant of improving search, elegantly adapted to the peculiarities of LP in standard form
- Every step of the simplex visits an extreme point of the LP feasible domain

Standard Display:

$$
\begin{aligned}
\max\quad & x_1 + x_2\\
\text{s.t.}\quad & -x_1 + x_2 \geq 0\\
& x_1 \leq 2\\
& x_2 \leq 3\\
& x_1, x_2 \geq 0
\end{aligned}
\quad\Longrightarrow
$$

	x_1	x_2	x_3	x_4	x_5	
max **c**	1	1	0	0	0	**b**
A	-1	1	-1	0	0	0
	1	0	0	1	0	2
	0	1	0	0	1	3

Getting Started!

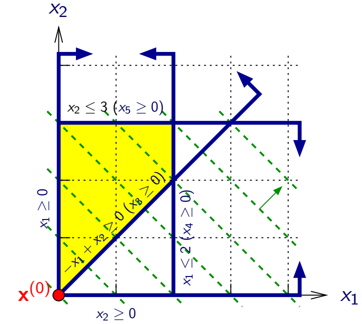
An improving search begins by choosing a starting feasible solution, and simplex requires an extreme point:

Initial Basic Solution

Simplex search begins at an extreme point of the feasible region (i.e., at a basic feasible solution to the model in standard form)

Example: Try with basic variables $\mathscr{B}^{(0)} = \{x_2, x_4, x_5\}$?

	x_1	x_2	x_3	x_4	x_5	
max **c**	1	1	0	0	0	**b**
A	-1	1	-1	0	0	0
	1	0	0	1	0	2
	0	1	0	0	1	3
$\mathscr{B}^{(0)}$	N	B	N	B	B	
$\mathbf{x}^{(0)}$	0	0	0	2	3	$\mathbf{c}^\mathsf{T}\mathbf{x}^{(0)} = 0$



Simplex Directions

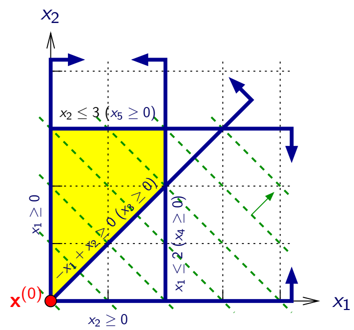
We want simplex to follow edge directions joining current extreme points to adjacent ones:

- Each edge direction follows a line determined by all but one of the active constraints at the current extreme point
- But we know that the active constraints at a basic feasible solution correspond to the nonnegativity constraints on nonbasic variables

Class Exercise: Starting from $\mathbf{x}^{(0)}$, how to follow the edge defined by:

1. the active constraint $-x_1 + x_2 = 0$?

2. the active constraint $x_1 = 0$?



Simplex Directions (cont'd)

There is one simplex direction for each nonbasic variable!

1. Let $\mathscr{B}$ be the current basic feasible solution, and consider the nonbasic variable $x_j \notin \mathscr{B}$:
$$\Delta x_j = 1, \quad \text{and} \quad \Delta x_k = 0, \quad \forall x_k \notin \mathscr{B}, k \neq j$$
2. We want to remain feasible as we move along an edge. What is the change in basic variables $x_k \in \mathscr{B}$ needed to preserve the equality constraints $\mathbf{Ax} = \mathbf{b}$?
$$\left.\begin{aligned} \mathbf{Ax} &= \mathbf{b} \\ \mathbf{A}(\mathbf{x} + \Delta\mathbf{x}) &= \mathbf{b} \end{aligned}\right\} \implies \mathbf{A}\Delta\mathbf{x} = \mathbf{0}$$

Constructing Simplex Direction

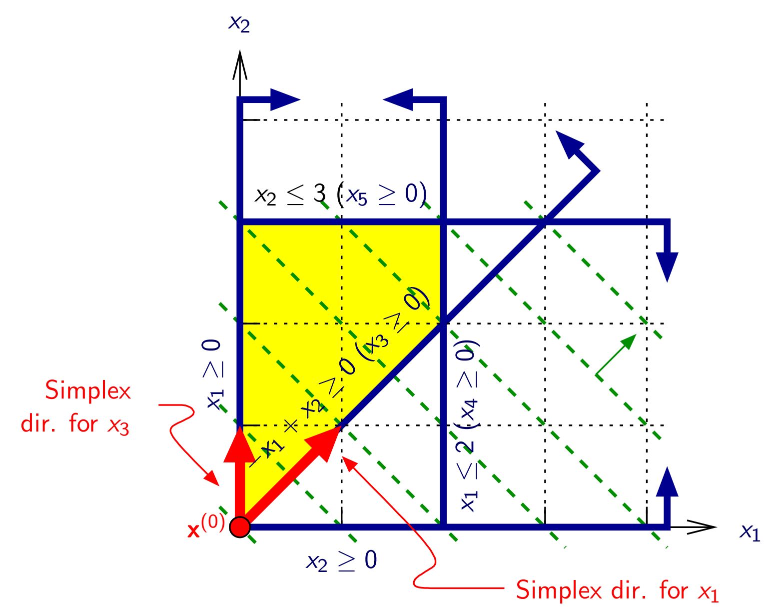
Simplex directions are constructed by increasing a single nonbasic variable, leaving other nonbasic unchanged, and computing the (unique) corresponding change in basic variables necessary to preserve equality constraints

Constructing Simplex Directions

Class Exercise: Calculate the simplex directions corresponding to the basic feasible solution $\mathscr{B}^{(0)} = \{x_2, x_4, x_5\}$ for the following standard-form LP:

	x_1	x_2	x_3	x_4	x_5	
max $\mathbf{c}$	1	1	0	0	0	$\mathbf{b}$
$\mathbf{A}$	-1	1	-1	0	0	0
	1	0	0	1	0	2
	0	1	0	0	1	3
$\mathscr{B}^{(0)}$	N	B	N	B	B	
$\mathbf{x}^{(0)}$	0	0	0	2	3	$\mathbf{c}^{\mathsf{T}}\mathbf{x}^{(0)} = 0$
$\Delta\mathbf{x}$ for x_1						
$\Delta\mathbf{x}$ for x_3						

Constructing Simplex Directions (cont'd)



Which Simplex Direction to Follow?

Our next task is to see whether any of the simplex directions improve the objective function $f(\mathbf{x}) = \mathbf{c}^{\mathsf{T}}\mathbf{x}$

- Moving along a simplex direction $\Delta\mathbf{x}$ for nonbasic x_j incurs a change in objective equal to
$$\bar{c}_j \triangleq \mathbf{c}^{\mathsf{T}}\Delta\mathbf{x}$$
- $\bar{c}_j$ is the so-called reduced cost

Improving Simplex Directions

The simplex direction $\Delta\mathbf{x}$ increasing nonbasic x_j is improving if:

- $\bar{c}_j > 0$, for a maximize problem
- $\bar{c}_j < 0$, for a minimize problem

Checking Improvement of Simplex Directions

Class Exercise: Determine which of the simplex directions computed previously are improving for the specified maximizing objective function:

	x_1	x_2	x_3	x_4	x_5	
max $\mathbf{c}$	1	1	0	0	0	$\mathbf{b}$
$\mathbf{A}$	-1	1	-1	0	0	0
	1	0	0	1	0	2
	0	1	0	0	1	3
$\mathscr{B}^{(0)}$	N	B	N	B	B	
$\mathbf{x}^{(0)}$	0	0	0	2	3	$\mathbf{c}^{\mathsf{T}}\mathbf{x}^{(0)} = 0$
$\Delta\mathbf{x}$ for x_1						$\bar{c}_1 =$
$\Delta\mathbf{x}$ for x_3						$\bar{c}_3 =$

How Long to Follow the Simplex Direction?

- For the next move, simplex can adopt any simplex direction $\Delta\mathbf{x}$ that improves the objective function
- The next issue is "How far?" — What step size λ in the direction $\Delta\mathbf{x}$?

The limit on step size λ can *only* come from violating a nonnegativity constraint. **Why?**

Maximum Acceptable Step

If any component is *negative* in improving simplex direction $\Delta\mathbf{x}$ at current basic solution $\mathbf{x}^{(k)}$, simplex search uses the maximum acceptable step,

$$\lambda = \min\left\{\frac{x_j^{(k)}}{-\Delta x_j} : \Delta x_j < 0\right\}$$

- **Why only negative components?**

Making the Next Move

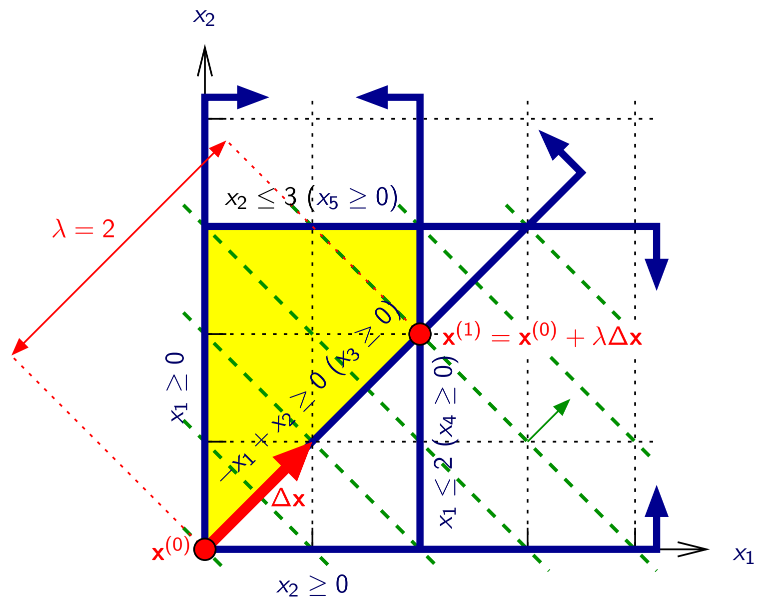
Class Exercise: Determine the maximum step and new solution in the selected improving simplex direction

	x_1	x_2	x_3	x_4	x_5	
$\mathscr{B}^{(0)}$	N	B	N	B	B	
$\mathbf{x}^{(0)}$	0	0	0	2	3	$\mathbf{c}^{\mathsf{T}}\mathbf{x}^{(0)} = 0$
$\Delta\mathbf{x}$	1	1	0	-1	-1	$\bar{c} = 2$
						$\lambda =$

Our **new solution** is:

$$\mathbf{x}^{(1)} \leftarrow \mathbf{x}^{(0)} + \lambda\Delta\mathbf{x} =$$

Making the Next Move (cont'd)



Updating the Basic Solution

To continue the algorithm, we need to find a new basic feasible solution — Active nonnegativity constraints tell us how:

Update

After each move of simplex search,

- the nonbasic variable generating the chosen simplex becomes basic
- any one of the (possibly several) basic variables fixing the step size becomes nonbasic

Class Exercise: Determine the new basic feasible solution in the previous exercise

	x_1	x_2	x_3	x_4	x_5
$\mathscr{B}^{(0)}$	N	B	N	B	B
$\mathscr{B}^{(1)}$					

Rudimentary Simplex Search for LP

- **Step 0: Initialization**
  - Choose any starting feasible basic solution $\mathscr{B}^{(0)}$, construct the starting point $\mathbf{x}^{(0)}$, and let index $k \leftarrow 0$.
- **Step 1: Simplex Directions**
  - Construct the simplex directions $\Delta\mathbf{x}$ associated with increasing each nonbasic x_j, and compute the corresponding reduced cost $\bar{c}_j = \mathbf{c}^\mathsf{T}\Delta\mathbf{x}$
  - If no simplex direction is improving, stop — current solution $\mathbf{x}^{(k)}$ is globally optimal
  - Otherwise, choose any improving simplex direction $\Delta\mathbf{x}$, and denote the entering basic variable x_e

Rudimentary Simplex Search for LP

- **Step 2: Step Size**
  - If there is no limit on feasible moves in simplex direction $\Delta\mathbf{x}$, stop — The model is unbounded.
  - Otherwise, choose the step size λ so that

$$\lambda = \min\left\{\frac{x_j^{(k)}}{-\Delta x_j} : \Delta x_j < 0\right\}$$

    and denote the leaving variable x_ℓ
- **Step 3: Update**
  - Compute the new solution point and basic solution:

$$\mathbf{x}^{(k+1)} \leftarrow \mathbf{x}^{(k)} + \lambda\Delta\mathbf{x}$$
$$\mathscr{B}^{(k+1)} \leftarrow \mathscr{B}^{(k)} \cup \{x_e\} \setminus \{x_\ell\}$$

  - Increment index $k \leftarrow k+1$ and return to Step 1

Determining the Optimum

Class Exercise: Continue the simplex search from the new basic feasible solution $\mathscr{B}^{(1)} = \{x_1, x_2, x_5\}$:

	x_1	x_2	x_3	x_4	x_5	
max **c**	1	1	0	0	0	**b**
A	-1	1	-1	0	0	0
	1	0	0	1	0	2
	0	1	0	0	1	3
$\mathscr{B}^{(1)}$	B	B	N	N	B	
$\mathbf{x}^{(1)}$						$\mathbf{c}^\mathsf{T}\mathbf{x}^{(1)} =$

$\Delta\mathbf{x}$ for ...

⋮

Generating a Starting Basic Feasible Solution

In most problems, we have to search for a starting basic feasible solution, before Simplex search can be applied.

LP standard form:

$$\min_{\mathbf{x}} \; z \triangleq \mathbf{c}^\mathsf{T}\mathbf{x}$$
$$\text{s.t.} \quad \mathbf{A}\mathbf{x} = \mathbf{b}$$
$$\mathbf{x} \geq \mathbf{0}$$

$m \triangleq$ number of main constraints

$n \triangleq$ number of decision variables

Basic Feasible Solution:

- $(n - m)$ elements of $\mathbf{x}$ are set to zero $\Rightarrow$ nonbasics
- remaining m elements are nonnegative **and** satisfy $\mathbf{A}\mathbf{x} = \mathbf{b}$ $\Rightarrow$ basics

Idea

Formulate an artificial LP model, the solution of which provides a basic feasible solution to the original LP

Artificial LP Model Formulation and Solution

Artificial LP

An artificial LP is constructed as one that minimizes the sum of constraint violation for the equality constraints in the original LP model

Step 1: Introduce new nonnegative artificial variables $x_{n+1}, \ldots, x_{n+m}$ such that:

$$\begin{array}{llllllll}
a_{1,1}x_1 + & \cdots & +a_{1,n}x_n & \pm x_{n+1} & & & = & b_1 \\
a_{2,1}x_1 + & \cdots & +a_{2,n}x_n & & \pm x_{n+2} & & = & b_2 \\
\vdots & \ddots & \vdots & & & \ddots & = & \vdots \\
a_{m,1}x_1 + & \cdots & +a_{m,n}x_n & & & \pm x_{n+m} & = & b_m
\end{array}$$

- Add the artificial variable x_{n+j} if $b_j \geq 0$
- Subtract the artificial variable x_{n+j} otherwise

Artificial LP Model Formulation and Solution

Step 2: Formulate the following artificial LP with $(n+m)$ variables:

$$\begin{array}{lllllll}
\min_{\mathbf{x}} & v \triangleq x_{n+1} + \cdots + x_{n+m} \\
\text{s.t.} & a_{1,1}x_1 + \;\cdots\; +a_{1,n}x_n & \pm x_{n+1} & & = & b_1 \\
& \vdots \quad \ddots \quad \vdots & & \ddots & = & \vdots \\
& a_{m,1}x_1 + \;\cdots\; +a_{m,n}x_n & & \pm x_{n+m} & = & b_m \\
& x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m} \geq 0
\end{array}$$

- The artificial LP is in standard form — **Why?**
- $\mathscr{B}^{(0)} \triangleq \{x_{n+1}, \ldots, x_{n+m}\}$ is a basic feasible solution — **Why?**

Step 3: Solve the auxiliary LP, starting from $\mathscr{B}^{(0)} \triangleq \{x_{n+1}, \ldots, x_{n+m}\}$

Artificial LP Model Formulation and Solution

Class Exercise: Formulate an artificial LP model to identify a starting basic feasible solution to the following LP model in standard form:

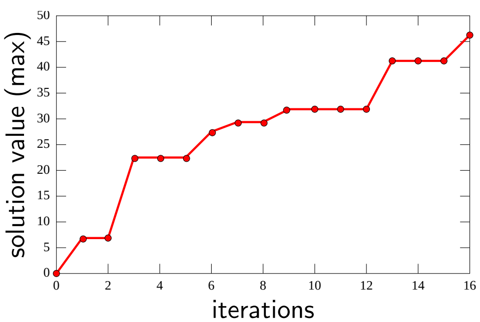
$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & -x_1 + x_2 - x_3 = 0 \\ & x_1 + x_4 = 2 \\ & x_2 + x_5 = 3 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

Two-Phase Simplex Search

1. **Phase I:**
   - Apply simplex search to the artificial LP model
2. **Infeasibility:**
   - If Phase I search terminates with a minimum having artificial sum $v > 0$, stop — The original LP model is infeasible
   - Otherwise, use the final Phase I basic solution to identify a starting feasible basic solution for the original LP model
3. **Phase II:**
   - Apply simplex search, starting from the identified basic feasible solution, to compute an optimal solution to the original standard-form LP model or prove that it is unbounded

Degeneracy

It is not always the case that a better extreme point is encountered at each iteration! With simplex search, progress at some iterations is typically interspersed with periods of no advance:



Degeneracy

A basic feasible solution to a standard-form LP is degenerate if nonnegativity constraints for some basic variables are active (i.e., more constraints are active than strictly needed to define an extreme point)

Degeneracy (cont'd)

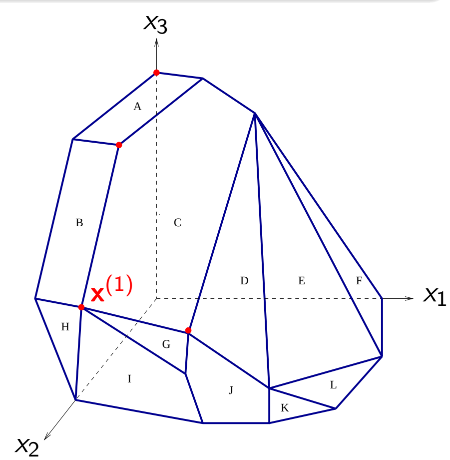
Multiple Choices of Basic Variables

In the presence of degeneracy, several sets of basic variables compute the same basic solution

Graphical Illustration:

- All 5 inequalities B, C, H, I and G are active at $\mathbf{x}^{(1)}$ — the corresponding slack variables have value $= 0$
- Any 3 of these 5 inequalities define extreme point $\mathbf{x}^{(1)}$
- Hence, $5 - 3 = 2$ basic variables have value $= 0$

The solution is degenerate!



Investigating the Effects of Degeneracy

Class Exercise: Consider the following LP model, with given starting basic feasible solution $\mathscr{B}^{(0)}$

1. Explain why the solution is degenerate
2. Show that 'Δx for x_1' is an improving simplex direction
3. Calculate the step length along that direction
4. Update the basic/nonbasic variables in $\mathscr{B}^{(1)}$ and calculate $\mathbf{x}^{(1)}$

	x_1	x_2	x_3	x_4	x_5	
max $\mathbf{c}$	1	1	0	0	0	$\mathbf{b}$
$\mathbf{A}$	-1	1	-1	0	0	0
	1	0	0	1	0	2
	0	1	0	0	1	3
$\mathscr{B}^{(0)}$	N	N	B	B	B	
$\mathbf{x}^{(0)}$	0	0	0	2	3	$\mathbf{c}^\mathsf{T}\mathbf{x}^{(0)} = 0$
$\Delta\mathbf{x}$ for x_1						$\bar{c}_1 =$
						$\lambda =$
$\mathscr{B}^{(1)}$						
$\mathbf{x}^{(1)}$						$\mathbf{c}^\mathsf{T}\mathbf{x}^{(1)} =$

Zero-Length Simplex Steps

- Simplex directions that decrease basic variables already $= 0$ in a degenerate solution may produce moves with steps $\lambda = 0$

Rules

When degenerate solutions cause the simplex algorithm to compute a step $\lambda = 0$,

1. the basic/nonbasic variables should be changed accordingly,
2. computations should be continued as if a positive step had been taken

- Simplex computations will normally escape a sequence of degenerate moves
  (changing basic representations eventually produces a direction along which positive progress can be achieved)

The Final Words on Simplex Search

Convergence with Simplex

If each iteration yields a positive step $\lambda > 0$, simplex search will stop after finitely many iterations, with either an optimal solution or an indication of unboundedness

$$\text{maximum number of extreme points} \triangleq \frac{n!}{(n-m)!m!}$$

This is a finite number, yet it can be very large!

Degeneracy and Cycling

- Cycling can occur with degenerate solutions: a sequence of degenerate moves may return to a basic/nonbasic configuration it has already visited!
- It is usually safe to assume that cycling will not occur in applied LP models, and thus that simplex search will converge finitely